

# ISOMETRIES OF TWO DIMENSIONAL HILBERT GEOMETRIES

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**ABSTRACT.** We prove that any isometry between two dimensional Hilbert geometries is a projective transformation unless the domains are interiors of triangles.

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*Dedicated to Pierre de la Harpe on his seventieth birthday.*

## 1. INTRODUCTION

The *Hilbert distance* between two points  $x$  and  $y$  in a bounded convex domain  $\Omega$  of  $\mathbb{R}^n$  is defined as

$$(1.1) \quad d(x, y) := \ln((x, y; \bar{x}, \bar{y})) := \ln \left( \frac{|\bar{y} - x|}{|\bar{y} - y|} : \frac{|\bar{x} - x|}{|\bar{x} - y|} \right),$$

where  $|u - v|$  denotes the usual Euclidean length between two points  $u$  and  $v$  in  $\mathbb{R}^n$ , and  $\bar{x}$  and  $\bar{y}$  are as on Fig. 1. It is well known, that the distance function  $d$  satisfies the standard requirements of a distance function, the only nontrivial point to check being the triangle inequality, see for example [7] or [5, §1]. This distance has been introduced by Hilbert in [7] and we refer to [6] for a presentation of both classic and contemporary aspects of Hilbert geometry<sup>1</sup>.

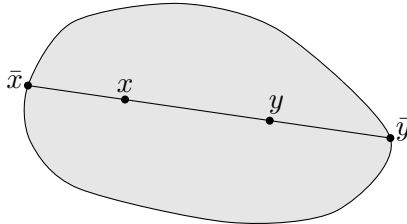
Recall that straight lines, convexity, and the cross ratio of four aligned points are invariant under projective transformations, this implies immediately that if  $f : \mathbb{RP}^n \rightarrow \mathbb{RP}^n$  is a projective transformation, then its restriction to  $\Omega$  defines an isometry  $f : \Omega \rightarrow f(\Omega)$ , with respect to the Hilbert distances in  $\Omega$  and  $f(\Omega)$ . (We consider  $\mathbb{R}^n$  as a subset of  $\mathbb{RP}^n$  by identifying it with an affine chart, the Hilbert metric inside  $\Omega$  does not depend on the choice of the affine chart.) The converse to this statement is not always true: some

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<sup>1</sup>Although we have assumed that  $\Omega \subset \mathbb{R}^n$  is a bounded convex domain, the Hilbert distance (1.1) is well defined for the more general class of *proper* convex domains. A convex domain in  $\mathbb{RP}^n$  is proper if it does not contain any full affine line. It is known that a convex domain is proper if and only if it is projectively equivalent to a bounded convex domain. For convenience we will therefore consider only bounded convex domains.

FIGURE 1. The points  $\bar{x}$  and  $\bar{y}$ 

special Hilbert geometries admit isometries which are not projective transformations. The simplest example is given by the simplex and is discussed in details in dimension 2 by Pierre de la Harpe in [5]. This author asked for a full description of all isometries in Hilbert geometry and a complete answer in finite dimension has recently been obtained by Cormac Walsh in [14]. Note also that the same author, together with Bas Lemmens, previously described all isometries of polyhedral Hilbert geometries in [10], while Bas Lemmens, Mark Roelands and Marten Wortel gave some partial results in infinite dimension in [9].

Our goal in this paper is to give a short proof of the following two dimensional result:

**Theorem 1.1.** *Let  $\Omega_1$  and  $\Omega_2$  be two bounded convex domains in the plane  $\mathbb{R}^2$  and  $d_1, d_2$  be the corresponding Hilbert metrics. Suppose that  $\Omega_1$  is not the interior of a triangle, then every isometry  $f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)$  is the restriction of a projective transformation of  $\mathbb{RP}^2$ .*

As mentioned above, this result is false if  $\Omega_1$  is the interior of a triangle. In that case  $(\Omega_1, d_1)$  is isometric to a Minkowski plane whose unit ball is a regular hexagon and its group of isometries is not difficult to describe, see [5]. Recall also that above theorem is a special case of the result of C. Walsh [14, Theorem 1.3]. For the case of quadrilaterals, the result is also proved by P. de la Harpe in [5, Proposition 4].

Our proof uses completely different methods from those in Walsh's paper. It is quite direct and only based on the description of metric geodesics in Hilbert geometry, together with a quite old and nontrivial result from line geometry which is due W. Prenowitz.

## 2. THE CASE OF STRICTLY CONVEX DOMAINS

It will be convenient to start with the case of a strictly convex domain. In fact we will prove the following result:

**Proposition 2.1.** *Assume that  $\Omega_1 \subset \mathbb{R}^n$  is a strictly convex domain, then every isometry  $f : (\Omega_1, d_1) \rightarrow (\Omega_2, d_2)$  is the restriction of a projective transformation of  $\mathbb{RP}^n$ .*

This result is proved in [5, Proposition 3], but we shall give a slightly more direct proof. The result has recently been extended in infinite dimension in [9, Theorem 1.2].

The proof is based on the structure of geodesics for the Hilbert distance. It is easy to check from the definition of the Hilbert distance that if three points  $x, y, z \in \Omega$  are aligned and  $z \in [x, y]$ , then  $d_1(x, y) = d_1(x, z) + d_1(z, y)$ . In other words the intersection of Euclidean straight lines with  $\Omega_1$  are geodesics for the Hilbert metric. Furthermore, the following fact is classical (see [5, Proposition 2] or [11, Theorem 12.5]):

**Lemma 2.2.** *Let  $p$  and  $q$  be two points on the boundary of  $\Omega_1$ , and suppose that at least one of them is an extreme point of  $\Omega_1$ . Then the open interval  $(p, q)$  is the unique geodesic between any pair of its point, that is if  $x, y \in (p, q)$  and  $z \in \Omega_1$ , then  $d_1(x, y) = d_1(x, z) + d_1(z, y)$  if and only if  $z \in [x, y]$ .*

**Proof of Proposition 2.1.** Let  $f : \Omega_1 \rightarrow \Omega_2$  be an isometry for the Hilbert distances, where  $\Omega_1 \subset \mathbb{R}^n$  is strictly convex. From the previous Lemma, it then follows that the affine segment  $[x, y]$  between two points  $x, y \in \Omega_1$  is the unique geodesic joining these two points. Since  $f$  is an isometry, there is also a unique geodesic joining the images  $f(x)$  and  $f(y)$  in  $\Omega_2$  and because the Euclidean segment  $[f(x), f(y)] \subset \Omega_2$  is known to be geodesic we conclude that  $f$  maps the segment  $[x, y] \subset \Omega_1$  to the segment  $[f(x), f(y)] \subset \Omega_2$ . Since  $x$  and  $y$  are arbitrary points in  $\Omega_1$ , we conclude that  $f$  is a *local collineation*, that is a mapping sending Euclidean segments to Euclidean segments. The conclusion now follows from the local version of the fundamental theorem of projective geometry (see e.g. [13, Lemma 4]), which states that any local collineation defined in some open set of the real projective space  $\mathbb{RP}^n$  is the restriction of a projective transformation. □

### 3. PROOF OF THE MAIN THEOREM

The proof of Theorem 1.1 will be based on a 1935 result of Prenowitz [12] which generalizes the fundamental theorem of projective geometry in dimension 2. We will need the following definitions.

**Definitions 3.1.** Let  $U$  be a plane domain, that is an open connected nonempty subset of  $\mathbb{R}^2$ . By a *line in  $U$*  we mean a connected component of the intersection of a Euclidean straight line with  $U$ . A *family of lines* in  $U$  is a partition of  $U$  by lines, that is a collection of lines in  $U$  such that each point of  $U$  lies on exactly one line of the collection. If all lines in a family concur to a common point  $A$ , the family is called a *pencil with pole  $A$* . A (linear)  *$n$ -web* in  $U$  is a set of  $n$  families of lines on  $U$  such that no two families have a common line.

Figure 2 shows a pencil with pole  $A$  in the domain  $U$ . By taking the pencils through  $n$  pairwise distinct poles  $A_1, \dots, A_n \notin U$  we obtain an  $n$ -web in any

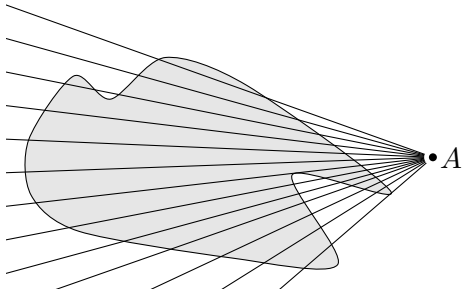


FIGURE 2. A pencil of lines covering a plane domain.

subdomain  $U' \subset U$  disjoint from any line through a pair of distinct points  $A_i, A_j$ .

**Theorem 3.2** (Prenowitz 1935). *A one to one continuous map defined in a plane domain that carries a 4-web into a 4-web is the restriction of a projective transformation.*

Recall that by Brouwer's theorem, an injective continuous map defined in a domain of  $\mathbb{R}^n$  is a homeomorphism onto its image. The above result is proved in [12]; a much simpler proof is given in [8] assuming the map is a diffeomorphism. Some generalization in higher dimensions are given in [1]. The following corollary will be useful in the proof:

**Corollary 3.3.** *Let  $f : U \rightarrow \mathbb{R}^2$  be a one to one continuous map defined in a domain  $U \subset \mathbb{R}^2$  and let  $A_1, \dots, A_5 \in \mathbb{R}^2$  be five pairwise distinct points. Assume that  $f$  maps the intersection of any line through  $A_j$  with  $U$  to a straight line ( $1 \leq j \leq 5$ ). Then  $f$  is the restriction of a projective transformation.*

*Proof.* There are 10 lines through any pair of the points  $A_j$  and the pairwise intersections of those 10 lines determine (at most) 20 points<sup>2</sup>. Let us denote by  $\mathcal{I}$  this set and call it the set of *intersection points*. For any point  $X \in U \setminus \mathcal{I}$ , at least four of the directions  $\overrightarrow{XA_j}$  are mutually distinct and this property holds in a neighborhood  $V$  of  $X$ . The pencils with poles the corresponding four points  $A_j$  form a 4-web in  $V$ , see Figure 3, which is mapped by  $f$  to a 4-web in  $f(V)$ . By Theorem 3.2, we know that the restriction of  $f$  to  $V$  is the restriction of a projective transformation. By real analyticity, two projective transformations that coincide on an open subset coincide everywhere. Since  $U \setminus \mathcal{I}$  is connected the restriction of  $f$  to  $U \setminus \mathcal{I}$  is a projective transformation and since  $\mathcal{I}$  is finite,  $f$  is a projective transformation on the whole domain  $U$  by continuity.  $\square$

<sup>2</sup>10 distinct lines in a projective plane define  $\binom{10}{2} = 45$  intersection points counted with multiplicity, the 5 points  $A_j$  have multiplicity 6.

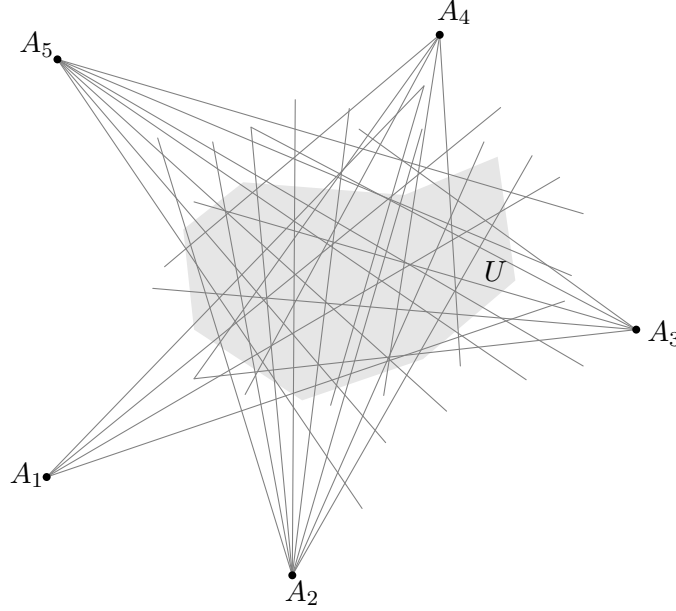


FIGURE 3. A polygonal region  $U$  covered by 5 pencils. Corollary 3.3 states that a homeomorphism defined in  $U$  carrying all those lines to some families of lines is a projective transformation.

**Proof of Theorem 1.1.** Recall that we assumed that the bounded convex domain  $\Omega_1 \in \mathbb{R}^2$  is not the interior of a triangle. We first assume that  $\Omega_1$  is also not a quadrilateral. Then, the boundary  $\partial\Omega_1$  contains at least five distinct extremal points  $A_1, A_2, A_3, A_4, A_5$ . Because the points  $A_j$  are extreme points of  $\Omega_1$ , Lemma 2.2 implies that each line through one of the point  $A_j$  intersects  $\Omega_1$  on a unique geodesic (for the Hilbert distance) between any of its pair of point. Since  $f$  is an isometry, it sends each line from the five pencils to a straight line in  $\Omega_2$  and it follows from Corollary 3.3 that  $f$  is the restriction of a projective transformation.

Suppose now that  $\Omega_1$  is a quadrilateral with vertices  $ABCD$ . The vertices are extreme points of  $\Omega_1$ , therefore, by Lemma 2.2, any line through a vertex defines a unique geodesic for the Hilbert distance and it is thus mapped on a line by the isometry  $f$ . The pencils with poles the four vertices form a 4-web in each connected component of the complement of the diagonals (these components are the triangles and  $DAM$ , where  $M$  is the intersection of the diagonals). Using Prenowitz' Theorem 3.2, we conclude that the restriction

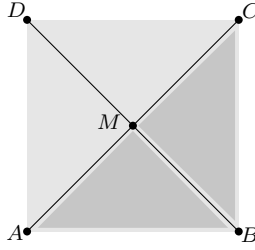


FIGURE 4. The restriction of  $f$  to the “dark-gray” triangles  $ABM$  and  $BCM$  is a projective transformation, and  $f$  sends  $AC$  to a straight line. Then, the image of  $ABC$  is a triangle and the restriction of  $f$  to it is a projective transformation.

of  $f$  to each of the triangles  $ABM$ ,  $BCM$ ,  $CDM$ ,  $DAM$  is a projective transformation.

Consider two adjacent such triangles, and consider the  $f$ -image of their union, see Fig 4. Since the restriction of  $f$  to each of these triangles is a projective transformation, the image of its union is two triangles. By continuity it has a common edge. Since the image of the line  $AC$  is a straight line, the closure of the image of the union of these triangles is a triangle. Furthermore, the map  $f$  sends any line through  $A$  or  $B$  to a line, we thus conclude that  $f$  restricted to the triangle  $ABC$  is a projective transformation (see also the Corollary in [12] page 567). Similarly, the restrictions of  $f$  to  $BCD$  and to  $CDA$  are projective transformations, which implies that the map  $f$  on the whole quadrilateral  $ABCD$  is the restriction of projective transformation as desired.

□

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